Dynamic Programming: Graph Algorithms

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Dynamic Programming: Algorithms for Graphs

- Most graph properties address optimisation goals, namely
  a. Shortest paths
  b. Minimum Spanning Trees
  c. Minimum Hamiltonian tours (Traveling Salesman)
  d. Minimum number of colours
- Some of these properties (e.g. **a** and **b**, but not **c** nor **d**), can be computed by polynomial algorithms.
- In this case, these algorithms follow a methodology dynamic programming, that is justified by Mathematical Induction on the Integers:
  - Once an optimal solution is obtained with n nodes, extend it to n+1 nodes.
- We will see two examples of this, in the following algorithms
  - Minimum Spanning Tree – **Prim's Algorithm**
  - Shortest Paths – **Floyd-Warshall's Algorithm**

Minimum Spanning Tree: Prim's Algorithm

- As discussed a spanning tree is a subset of a connected graph that has the topology of a tree and covers all nodes of the graph.
- It has many applications, namely to provide services to a number of sites (the nodes) that can be interconnected in several ways (by a graph), but using the a minimal number of connections that allow all sites to be reached, i.e. a single path connecting any two nodes.
- Among these spanning trees one is usually interested in Minimum Spanning Trees (MST) that minimise the sum of the costs of the arcs selected for the tree.
- There are many polynomial algorithms that may be used to compute these MSTs, the most common ones are the Kruskal's and the Prim's algorithms.
- Because of the similarities between the latter and the algorithm to check connectedness of a graph, we will address the **Prim's Algorithm**.

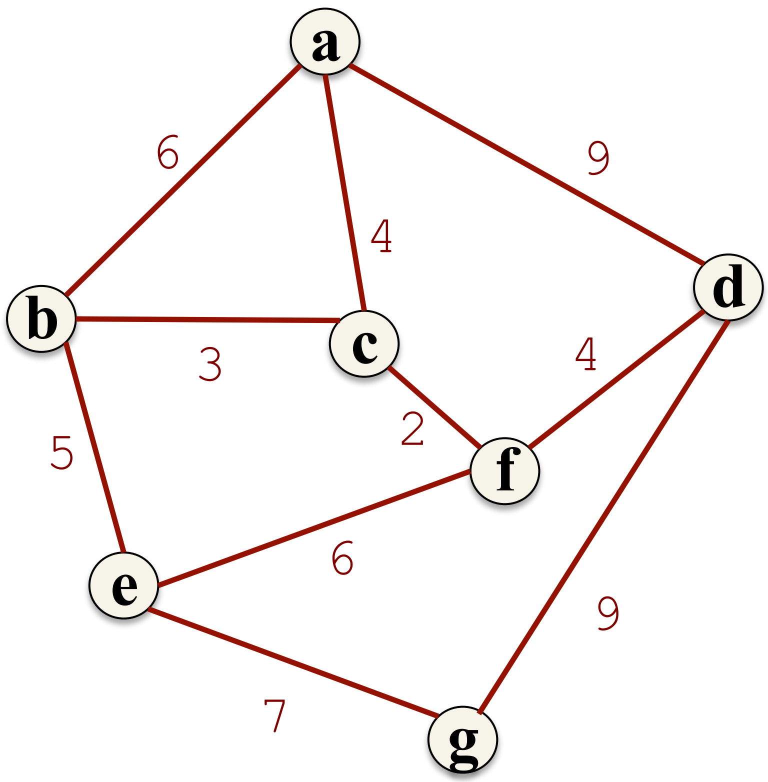
Minimum Spanning Tree: Prim's Algorithm

- The Prim´s algorithm is an example of **Dynamic Programming** that extends a MST with n nodes to n+1 nodes, with an **eager** selection of the new node (i.e. once the node is selected, the selection is not **backtracked**).
- The algorithm can be understood as a process of increasing the size of a current MST, starting with 1 node and ending with all the nodes, and specified as follows:

Maintain two sets of nodes: **In** and **Out**, where **In** is the set of nodes already included in a **current MST** and **Out** those not yet included.

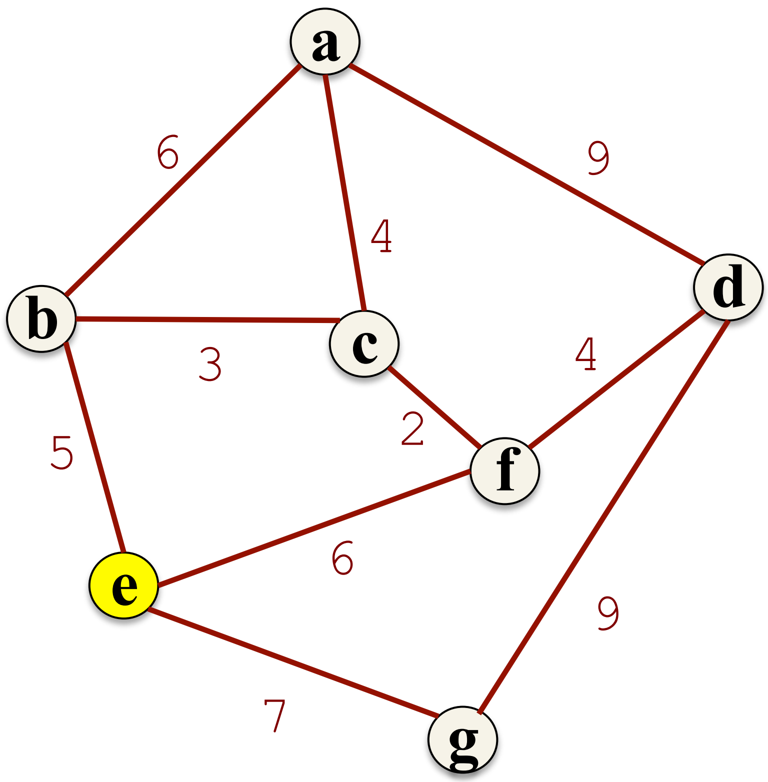
1. Select arbitrarily a node from the tree to initialise the **In** set, and put the others in the **Out** set;
2. While there are nodes in the **Out** set,
   - Find which node from the **Out** set has an arc of least cost to one connecting it to one of the nodes of the **In** set;
   - Transfer the node from the **Out** set to the **In** set and include the least cost arc in the **current MST**.

Minimum Spanning Tree: Prim's Algorithm




```
In = []
Fr = []
Out =
[a,b,c,d,e,f,g]
```

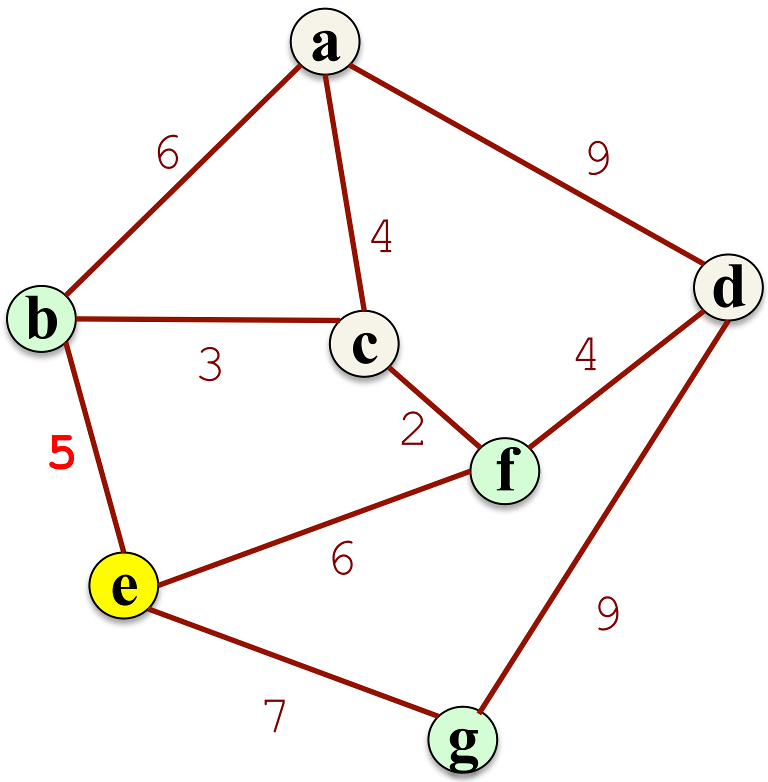
Minimum Spanning Tree: Prim's Algorithm



Chose an arbitrary node as the seed for the spanning tree

In = [e]
Fr = []
Out = [a,b,c,d,f,g]

Minimum Spanning Tree: Prim's Algorithm

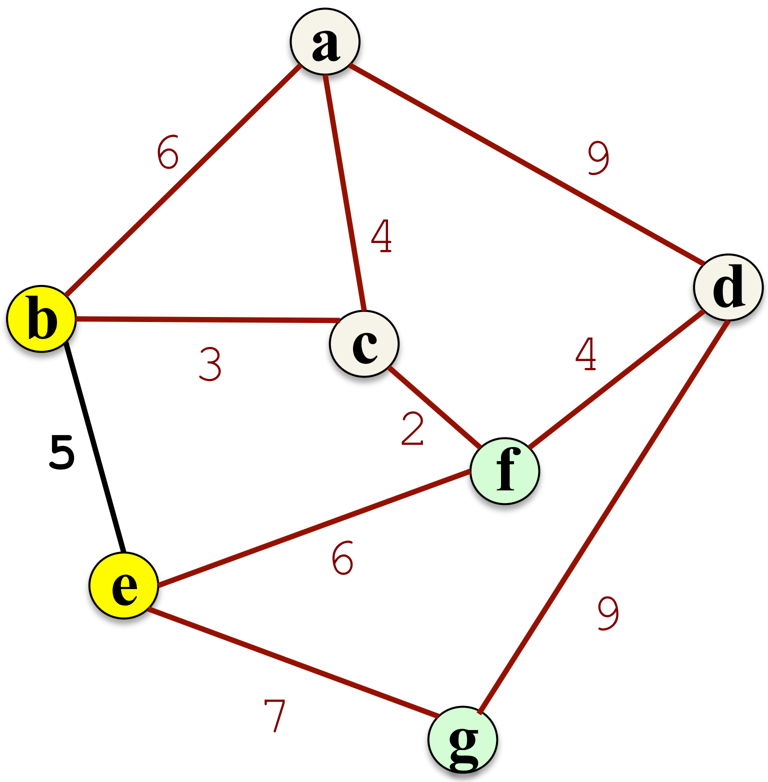


- Establish the nodes in the Frontier, and
- Find the shortest link between node in the In and Fr sets

<b,e,5>

```
In  = [e]
Fr  = [b,f,g]
Out = [a,c,d]
```

Minimum Spanning Tree: Prim's Algorithm

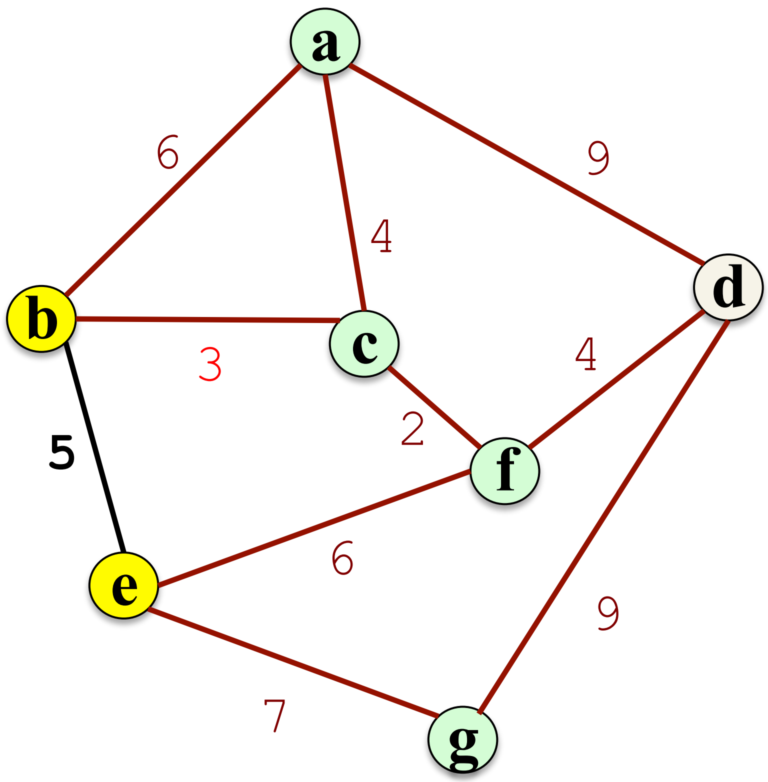


- Move to the In set, the selected node from the Frontier
- Add the arc to the spanning tree

<b,e,5>

```
In  = [b,e]
Fr  = [f,g]
Out = [a,c,d]
```

Minimum Spanning Tree: Prim's Algorithm

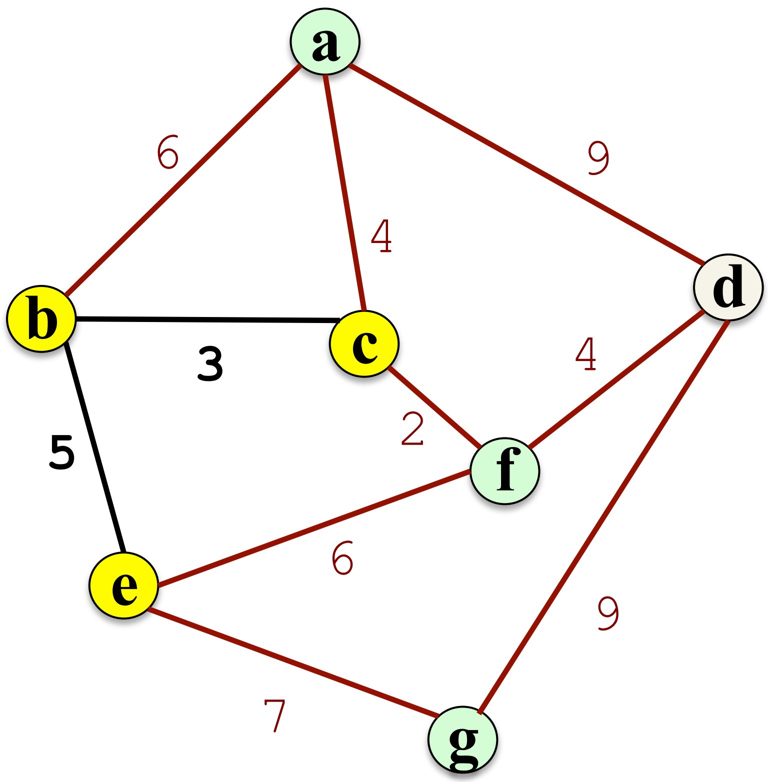


- Establish the nodes in the Frontier, and
- Find the shortest link between node in the In and Fr sets

<b,c,3>

```
In  = [b,e]
Fr  = [a,c,f,g]
Out = [d]
```

Minimum Spanning Tree: Prim's Algorithm

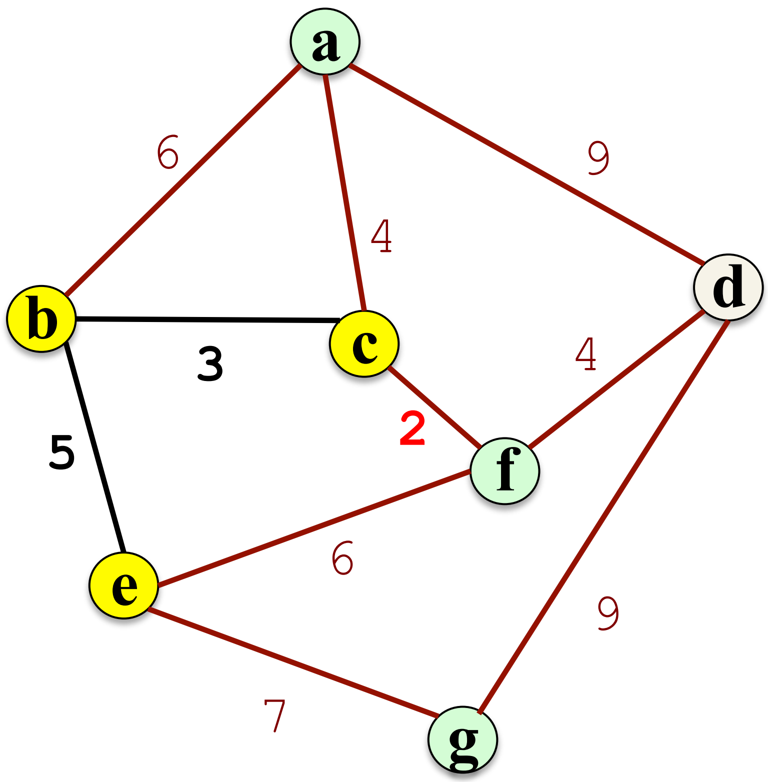


- Move to the In set, the selected node from the Frontier
- Add the arc to the spanning tree

<b,c,3>

```
In  = [c,b,e]
Fr  = [a,f,g]
Out = [c]
```

Minimum Spanning Tree: Prim's Algorithm

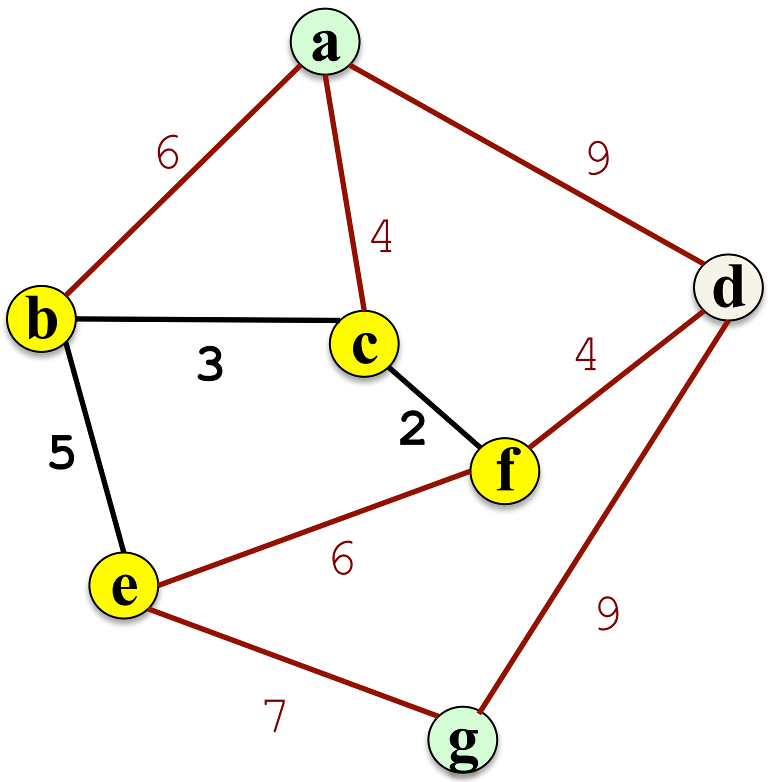


- Establish the nodes in the Frontier, and
- Find the shortest link between node in the In and Fr sets

<c,f,2>

```
In  = [c,b,e]
Fr  = [a,f,g]
Out = [d]
```

Minimum Spanning Tree: Prim's Algorithm

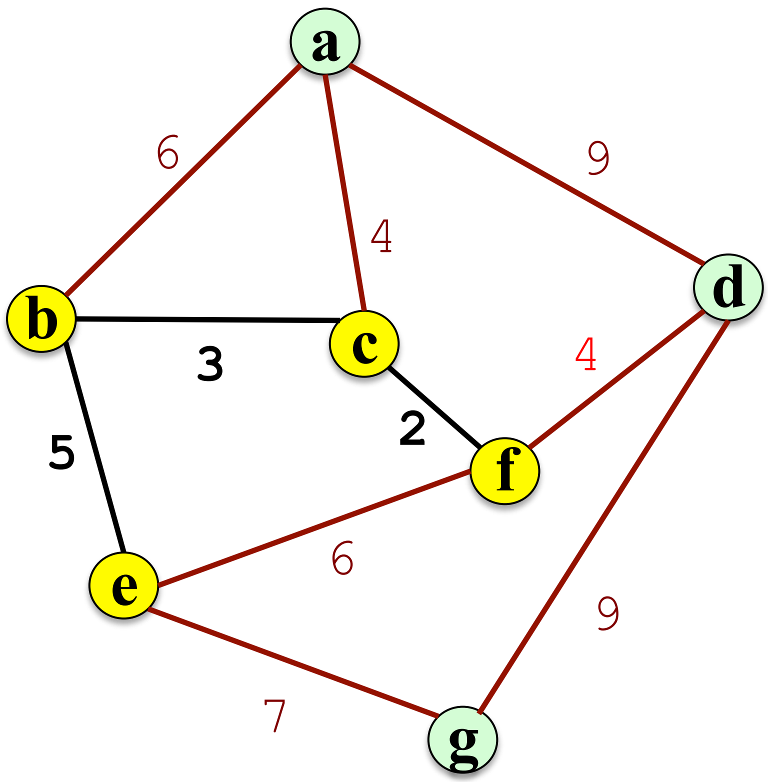


- Move to the In set, the selected node from the Frontier
- Add the arc to the spanning tree

<c,f,2>

```
In  = [f,c,b,e]
Fr  = [a,g]
Out = [c]
```

Minimum Spanning Tree: Prim's Algorithm

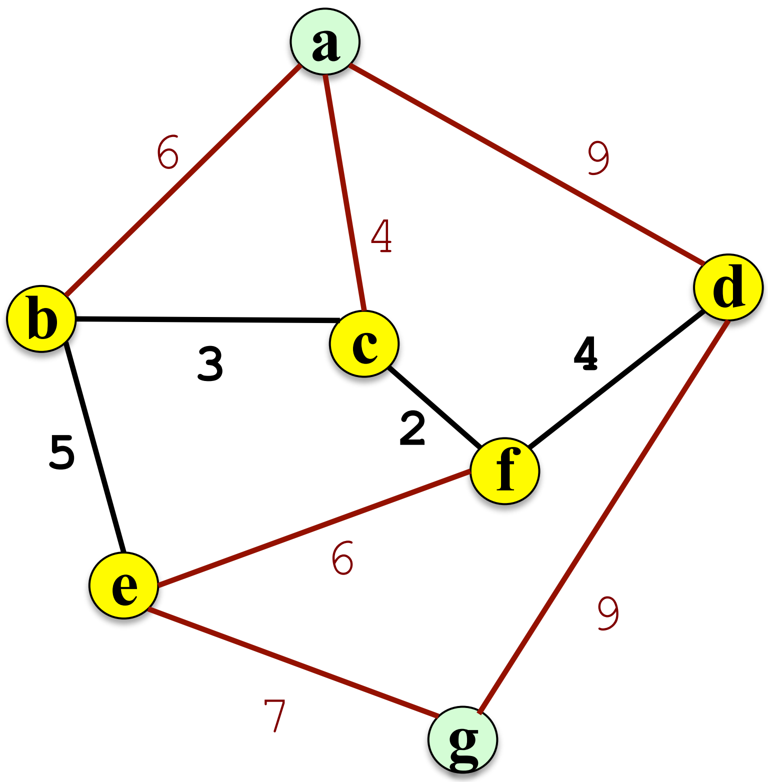


- Establish the nodes in the Frontier, and
- Find the shortest link between node in the In and Fr sets

<f,d,4>

```
In  = [f,c,b,e]
Fr  = [d,a,g]
Out = []
```

Minimum Spanning Tree: Prim's Algorithm

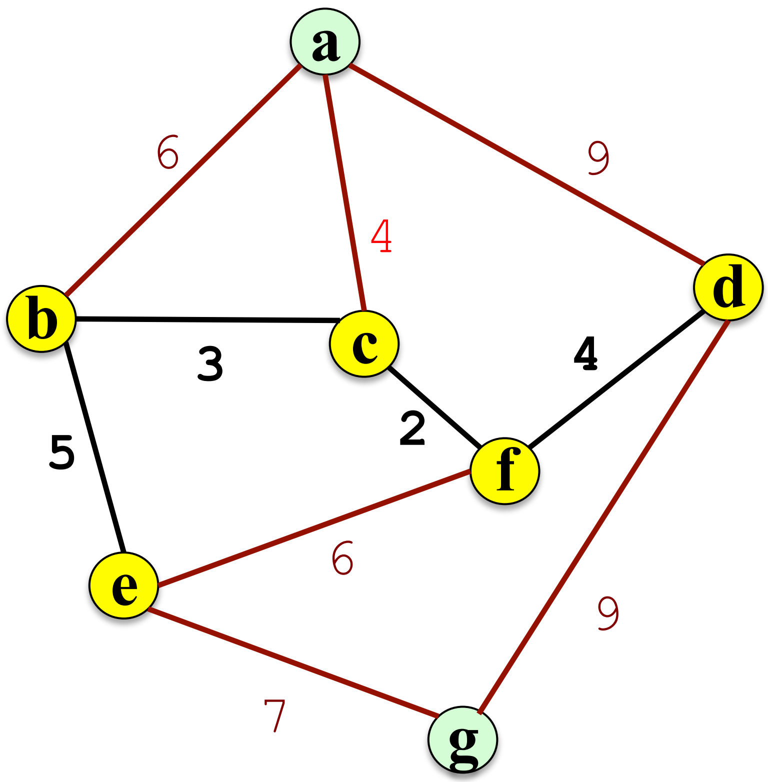


- Move to the In set, the selected node from the Frontier
- Add the arc to the spanning tree

<f,d,4>

```
In  = [d,f,c,b,e]
Fr  = [a,g]
Out = []
```

Minimum Spanning Tree: Prim's Algorithm

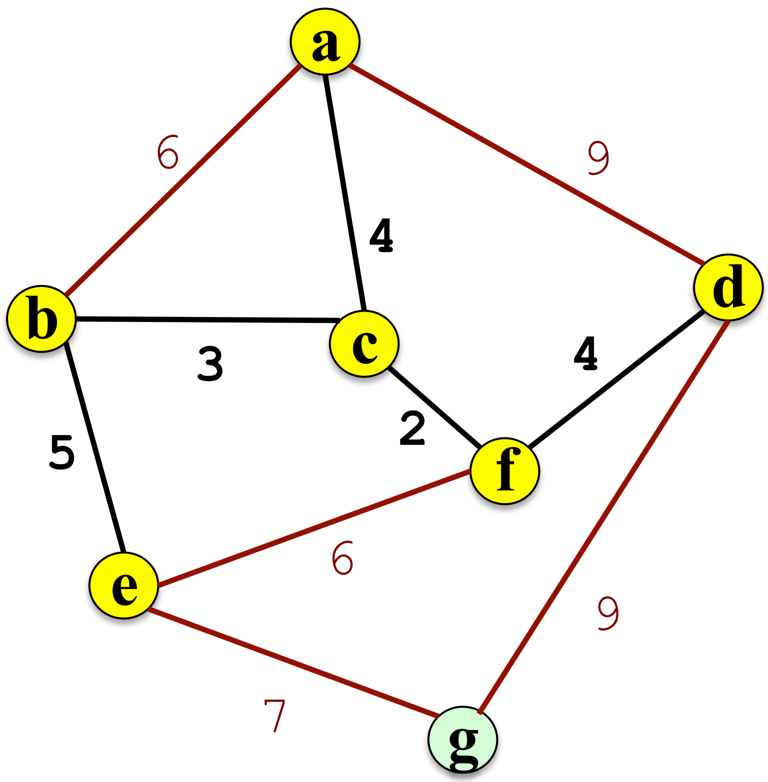


- Establish the nodes in the Frontier, and
- Find the shortest link between node in the In and Fr sets

<c,a,4>

```
In  = [d,f,c,b,e]
Fr  = [a,g]
Out = []
```

Minimum Spanning Tree: Prim's Algorithm

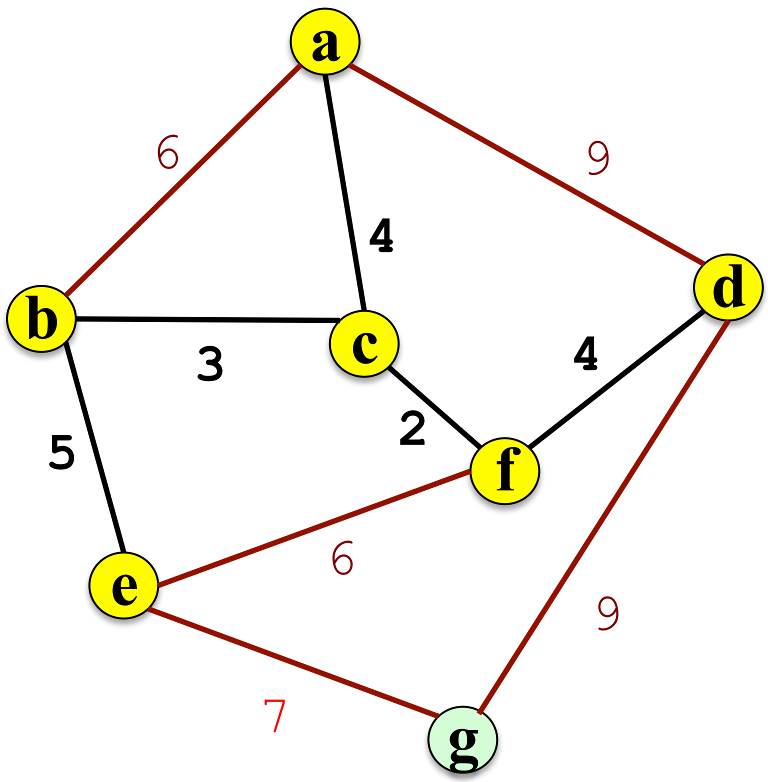


- Move to the In set, the selected node from the Frontier
- Add the arc to the spanning tree

<f,d,4>

```
In  = [a,d,f,c,b,e]
Fr  = [g]
Out = []
```

Minimum Spanning Tree: Prim's Algorithm

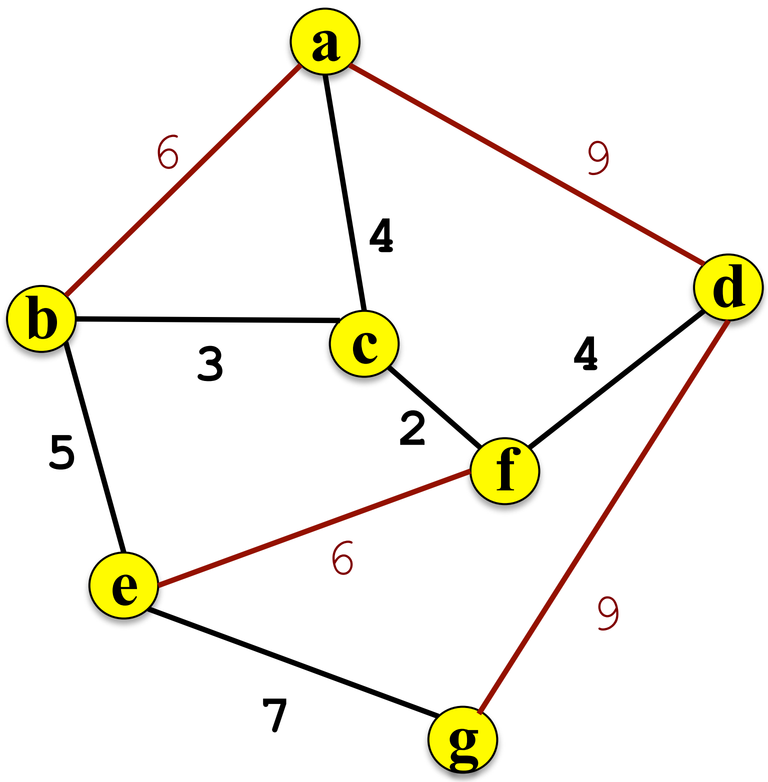


- Establish the nodes in the Frontier, and
- Find the shortest link between node in the In and Fr sets

<e,g,7>

```
In  = [a,d,f,c,b,e]
Fr  = [g]
Out = []
```

Minimum Spanning Tree: Prim's Algorithm

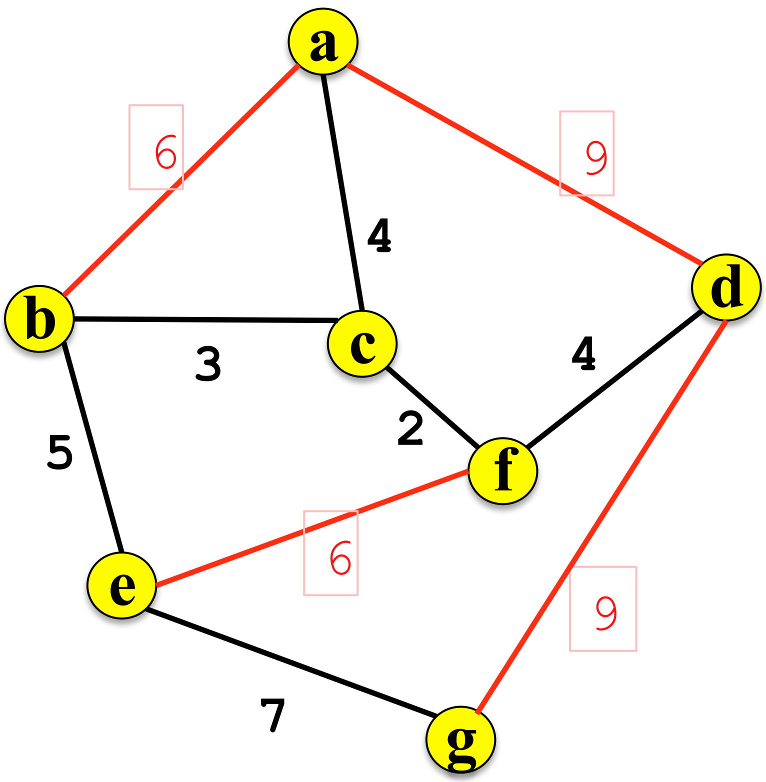


- Move to the In set, the selected node from the Frontier
- Add the arc to the spanning tree

<e,g,7>

```
In  = [g,a,d,f,c,b,e]
Fr  = []
Out = []
```

Minimum Spanning Tree: Prim's Algorithm



- Return the Spanning Tree

Minimum Spanning Tree: Prim's Algorithm

- Several variants can be used in the implementation of the Prim's algorithm, using appropriate data structures that make it more efficient. Here we present a naïf implementation that nonetheless is sufficient for relatively large graphs.

1. Select arbitrarily a node from the Graph to initialise the **In** set, and put the others in the **Out** set; Initialise the tree to "empty" – no arcs, i.e. arcs with cost Inf.
2. While there are nodes in the **Out** set
   - Find which node from the **Out** set has an arc of least cost to one connecting it to one of the nodes of the **In** set;
   - Transfer the node from the **Out** set to the **In** set and include the least cost arc in the **current MST**.

```
function [T] = prim(G);
   n = size(G,1);
   T = ones(n,n)*Inf;  In = [1]; Out = 2:n;
   while length(Out) > 0
      ...
      T = ...; In = ...; Out = ...
   endwhile
endfunction
```

Properties of Graphs

- The core of the algorithm is to find the arc with least cost connecting an arc between node of the **In** and **Out** sets (implemented as vectors).
- This can be performed with a standard search for a minimum value in a matrix, but in this case, restricted to indices of nodes in the **In** and **Out** sets.
- Additionally, the position **p** of the node in the **Out** vector is stored, to make it easy to remove it from the **Out** set.
- Finally, the arc is added to **T**, the current MST, and the **In** and **Out** sets updated.

```
minArc = Inf;
for i = 1: length(In)
   for j = 1:length(Out)
      if G(In(i),Out(j)) < minArc
         minArc = G(In(i),Out(j));
         v1 = In(i); v2 = Out(j);
         p = j;
      endif;
   endfor;
endfor;
T(v1,v2) = G(v1,v2); T(v2,v1) = G(v2,v1)
In = [v2,In]; Out = [Out(1:p-1),Out(p+1:end)]
```

Properties of Graphs

- The complete algorithm is shown below:

```
function [T] = prim(G);
   n = size(G,1);
   T = ones(n,n)*Inf;
   In = [1]; Out = 2:n;
   while length(Out) > 0
      minArc = Inf;
      for i = 1: length(In)
         for j = 1:length(Out)
            if G(In(i),Out(j)) < minArc
               minArc = G(In(i),Out(j));
               v1 = In(i); v2 = Out(j); p = j;
            endif;
         endfor;
      endfor;
      T(v1,v2) = G(v1,v2); T(v2,v1) = G(v2,v1)
      In = [v2,In];
      Out = [Out(1:p-1),Out(p+1:end)]
   endwhile
endfunction
```

Minimum Spanning Tree: Prim's Algorithm

- It is easy to prove, by induction, that the algorithm is correct. If T is an MST with least cost with **n** nodes, adding to it the least cost arc will make it an MST with least cost with n+1 nodes (adding any other arc would lead to a higher cost spanning tree).
- As to the worst cost complexity of the algorithm, with this implementation, we notice that the while loop is executed **n-1** times (**n** is the number of nodes of the graph, **|V|**).
- Finding the minimal cost arc requires two nested loops over ranges with **k** and **n-k** values, that is at most $\mathbf{n^2/4}$ executions (for **k = n/2**) of the body of the loop

```
for i = 1: length(In)
   for j = 1:length(Out)
      …
   endfor;
endfor
```

- All operations in the loop are "basic", and so the complexity of this implementation of the Prim's algorithm is $O(n*n^2/4)$ i.e. $\mathbf{O(|V|^3)}$.
- **Note**: Implementations with priority queues and other advanced data structures have better complexity, namely **O(|E|+Vlog|V|)**.

Shortest Paths – Floyd-Warshall's Algorithm

- There are many algorithms for finding shortest paths between nodes of weighted graphs. They include algorithms to find one shortest path between two nodes , like the Dijskstra algorith, or to find all shortest paths between any two nodes of the graph, namely the **Floyd-Warshall's (FW) algorithm**.
- As the previous one, the **FW** algorithm explores dynamic programming in the following way:
- If a shortest path is considered between any two nodes, considering all paths through a **List** of **In** nodes with **n** nodes, this shortest paths can be updated by extending the list of In nodes with an extra node.
- Starting with an empty List, and including one node at a time, the final results is the set of shortest paths between any two nodes.

Shortest Paths – Floyd-Warshall's Algorithm

- The algorithm can thus be specified as follows:
- Initialise a matrix S of shortest paths with the adjacency matrix, that is considers the direct distances between any two nodes.
  - On iteration one, update S, by considering all indirect paths passing through node 1.
  - On iteration 2, update S, by considering all indirect paths passing through nodes 1 and 2.

    …
  - On iteration n, update S, by considering all indirect paths passing through nodes 1,2..n, i.e. all paths.
- After this last iteration the set of all shortest paths between all nodes is stored in matrix S.
- Notice that this algorithm only computes the paths with shortest distance between any two nodes but does not return what these paths are.
  - In fact, a small addition to the algorithm allows the paths to be reconstructed.

Shortest Paths – Floyd-Warshall's Algorithm

- The implementation of this algorithm is shown now:

```
Function S = floyd(M)
   S = M;
   n = size(S,1);
   for k = 1:n
      for i = 1:n
         for j = 1:n
            if S(i,k) + S(k,j) < S(i,j)
               S(i,j) = S(i,k) + S(k,j);
            endif
         endfor
      endfor
   endfor
endfunction
```

- The external for loop guarantees that all paths passing through nodes 1..k have been considered previously.
- The shortest paths are updated by considering the triangular inequality, with paths passing through node n+1.

Shortest Paths – Floyd-Warshall's Algorithm

- The correction of the algorithm can be proved by induction on the number of nodes considered in indirect paths (left as exercise).
- As to the complexity, it is easy to see that the algorithm requires 3 nested or loops of size n, with basic operaton in the body,

```
for k = 1:n
   for i = 1:n
      for j = 1:n
         if S(i,k) + S(k,j) < S(i,j)
            S(i,j) = S(i,k) + S(k,j);
         endif
      endfor
   endfor
endfor
```

- The complexity of the algorithm is thus **O(|V|^3).**
- Notice that algorithms to compute shortest paths between 2 nodes, like the Dijskstra algorithm have complexity **O(|V|^2)**, but only consider a pair (not all) of nodes.

Path Reconstruction – Floyd-Warshall's Algorithm

- The previous algorithm does not provide the shortest paths between any two nodes, but rather the shortest distances of any path between the nodes.
- Nevertheless, these paths may be easily reconstructed if the initial arc of any shortest path between two nodes is recorded in a matrix **Next** (for next node).
- For every pair **<i,j>** the matrix is initialised with j, i.e. it assumes a direct path from I to j with no intermediate nodes.

```
for i = 1:n,
   for j = 1:n
      Next(i,j) = j;
   endfor,
endfor
```

- In the inner loop of the FW algorithm, if a new shortest path is found, the new leading arc is updated accordingly

```
if S(i,k) + S(k,j) < S(i,j)
   S(i,j) = S(i,k) + S(k,j);
   Next(i,j) = Next(i,k);
endif
```

Path Reconstruction – Floyd-Warshall's Algorithm

- The extended FW algorithm is shown below, returning the Next matrix.

```
function [S,Next] = floyd(M)
   S = M;
   n = size(S,1);
   for i = 1:n,
      for j = 1:n
         Next(i,j) = j;
      endfor,
   endfor
   for k = 1:n
      for i = 1:n
         for j = 1:n
            if S(i,k) + S(k,j) < S(i,j)
               S(i,j) = S(i,k) + S(k,j);
               Next(i,j) = Next(i,k);
            endif
         endfor
      endfor
   endfor
endfunction
```

Path Reconstruction – Floyd-Warshall's Algorithm

- Once the matrix Next is returned the path between any two nodes can be obtained by following the trail indicated by matrix Next, as shown below

```
function P = path(u,v,Next)
   if N(u,v) == inf
      P = [];
      return;
   endif
   P = [u];
   while u != v
      u = Next(u,v)
      P = [P,u];
   endwhile
endfunction
```

- With this reconstruction technique, the complexity of the FW algorthm is not changed , and the paths are only computed when needed.